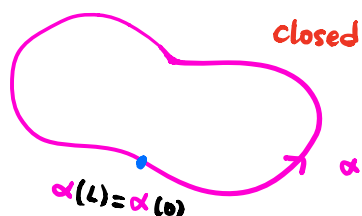


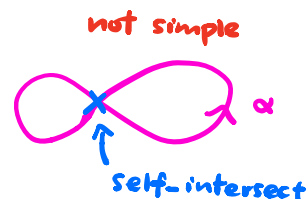
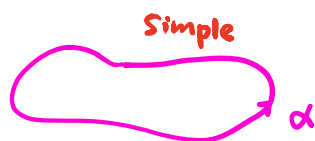
§ Some global theorems for plane curves

Defⁿ: A plane curve $\alpha: [0, L] \rightarrow \mathbb{R}^2$ is said to be

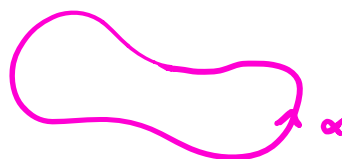
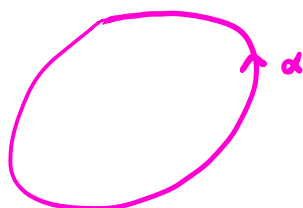
- **closed** if $\alpha^{(k)}(0) = \alpha^{(k)}(L)$ for $k = 0, 1, 2, \dots$



- **Simple** if α is 1-1 except possibly at $s = L$.



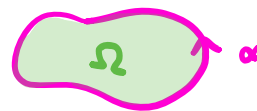
- **Convex** if $k > 0$ everywhere.



(A) Isoperimetric Inequality

For any **simple closed** curve in $\mathbb{R}^2$,

$$A \leq \frac{L^2}{4\pi}$$



$L = \text{Length}(\alpha)$

$A = \text{Area}(\Omega)$

"=" holds $\Leftrightarrow$ round circle

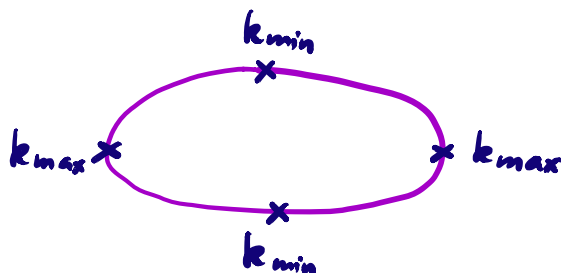
Proof: Exercise.

(B) Four Vertex Theorem

For any **simple closed (convex)** curve in $\mathbb{R}^2$,

$\exists$ at least **4** vertices (i.e. points where $k' = 0$)

ellipse has
exactly **4**
vertices.



Proof: omitted.

Note: It is easy to show that $\exists$ **2** vertices,

where k achieves its maximum and minimum. What is non-trivial is that there are at least 2 more!

(C) Theorem of Turning Tangents

For any **simple closed** curve α in $\mathbb{R}^2$, which is **positively oriented** (ie. **N** points inward)

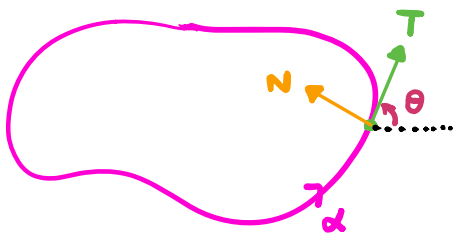
$$\int_{\alpha} k(s) ds = 2\pi$$

Proof: Let $\theta(s)$ be the angle from the positive x-axis to the unit tangent $T(s) = \alpha'(s)$, where α is p.b.a.l.

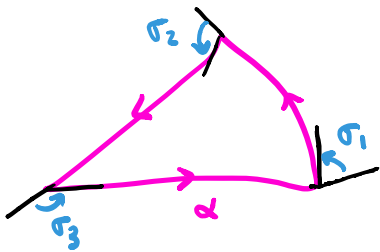
$$\theta'(s) = k(s)$$



$$\underbrace{\theta(L) - \theta(0)}_{= 2\pi} = \int_0^L k(s) ds$$



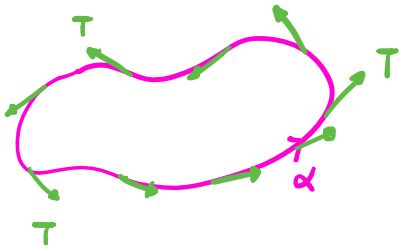
Non-smooth version:



$$\int_{\alpha} k(s) ds + \sum_i \sigma_i = 2\pi$$

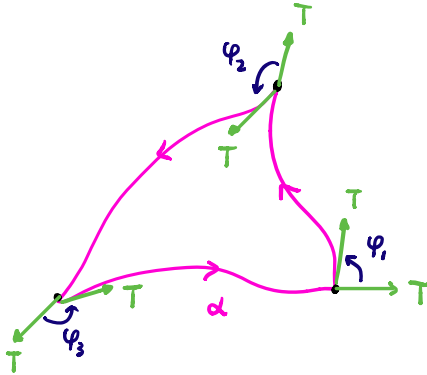
§ The Gauss-Bonnet Theorems

Recall: Theorem of Turning Tangents



$$\int_{\alpha} k(s) ds = 2\pi$$

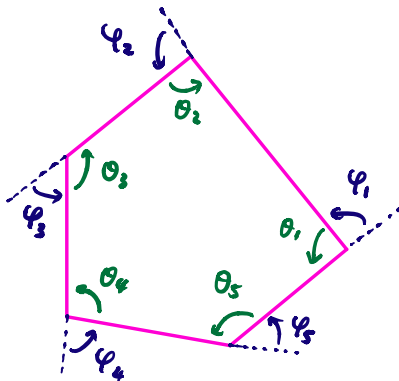
It also works for "piecewise smooth" curves:



$$\int_{\alpha} k(s) ds + \sum_i \varphi_i = 2\pi \quad (*)$$

Special case: When all the smooth pieces are **geodesics**

$(*) \Rightarrow$ exterior angle sum of polygons $= 2\pi$



interior angle sum of n -gon $= (n-2)\pi$

i.e. $\sum_{i=1}^n \varphi_i = 2\pi$, $\sum_{i=1}^n \theta_i = (n-2)\pi$.

Question: What about surfaces? Is the "total curvature" of a surface a constant?

Gauss Bonnet Theorem I:

Let S be a compact orientable surface without boundary.

Then

$$\int_S K \, da = 2\pi \chi(S)$$

where $\chi(S)$ is the Euler characteristics of S

given by the formula $\chi(S) = 2 - 2g$.

$\uparrow$ genus of S

Examples:



...



genus :

0

1

2

...

g

χ :

2

0

-2

...

$2-2g$



> 0

$= 0$

< 0

Note: The Gauss-Bonnet Theorem is a beautiful result
 Since it relates "geometry" on L.H.S. to "topology"
 on R.H.S.

There is a version for surfaces with boundary.

Gauss-Bonnet Theorem II:

Let S be a compact orientable surface with boundary.

Then,

$$\int_S K da + \int_{\partial S} k_g ds = 2\pi \chi(S)$$

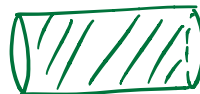
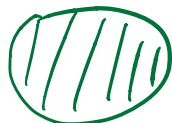
$\partial S \leftarrow \text{positively oriented}$

where $\chi(S) = 2 - 2g - \gamma$

$g = \text{genus}$

$\gamma = \# \text{ of boundaries}$

Example:



$g :$ 0

0

0

$\gamma :$ 1

2

3

$\chi :$ 1

0

-1



> 0



$= 0$



< 0

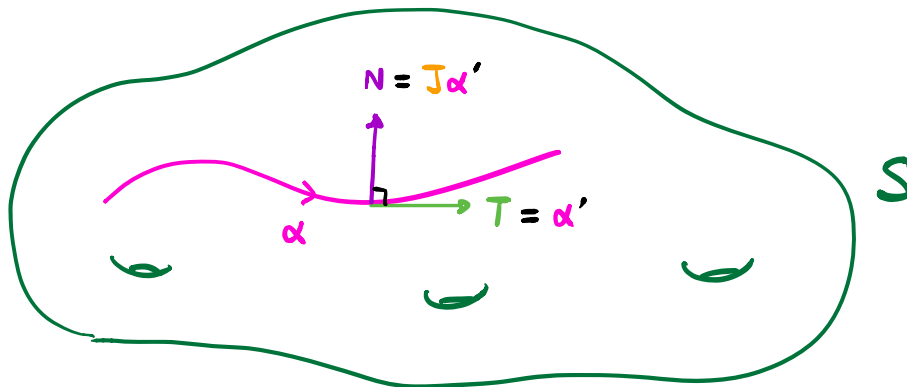
Definition: Let S be an oriented surface, then

$\exists J : T_p S \rightarrow T_p S$ rotation by 90° on each tangent plane.

If $\alpha : I \rightarrow S$ is a curve p.b.a.l.,

we define the geodesic curvature of α

$$k_g := \langle \nabla_{\alpha'} \alpha', J \alpha' \rangle$$



Remark: (1) If $S = \text{plane}$ (with standard orientation)

then k_g agrees with the usual curvature k for plane curves.

(2) Switch the orientation of α changes the sign of k_g

(3) α geodesic $\Leftrightarrow k_g \equiv 0$

(i.e. geodesic curvature measures how much is the curve deviated from a geodesic.)

There is also a version of Gauss-Bonnet Theorem which allows the boundary to be only "piecewise smooth":

$$\int_S K da + \int_{\partial S} k_g ds + \underbrace{\sum_i \varphi_i}_{\text{sum of exterior angles}} = 2\pi \chi(S)$$

Gauss-Bonnet Theorem III

sum of exterior angles.

§ Applications of Gauss-Bonnet Theorems

(I) Any compact orientable surface S without boundary and $K > 0$ everywhere is homeomorphic to a sphere.

Proof: By Gauss-Bonnet I,

$$\int_S K da = 2\pi \chi(S)$$

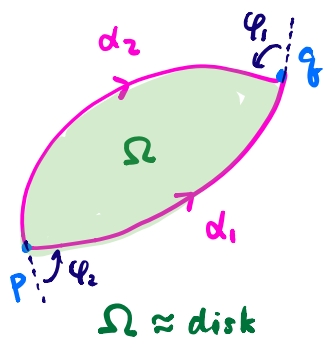
Since $K > 0$ everywhere, the L.H.S. > 0 , hence

$$\chi(S) > 0 \Rightarrow S \approx \text{sphere}.$$

_____ $\square$

(II) If S is a surface with $K \leq 0$ everywhere,
 then any two geodesics α_1, α_2 starting at the
 same point $p \in S$ cannot meet again at some $q \in S$
 s.t. α_1, α_2 together bounds a "disk".

Proof: By contradiction. Suppose not.



Apply Gauss-Bonnet III to Ω

$$\int_{\Omega} K \, da + \int_{\partial\Omega} k_g \, ds + \varphi_1 + \varphi_2 = 2\pi \chi(\Omega)$$

Since $\partial\Omega = \alpha_1 \cup \alpha_2$ are geodesics

$$\Rightarrow k_g \equiv 0.$$

Since $\Omega \approx \text{disk}$

$$\Rightarrow \chi(\Omega) = 1$$

Therefore,

$$\underbrace{\int_{\Omega} K \, da}_{\hat{0}} + \underbrace{\varphi_1}_{\hat{\pi}} + \underbrace{\varphi_2}_{\hat{\pi}} = 2\pi$$

Contradiction!

_____ $\square$

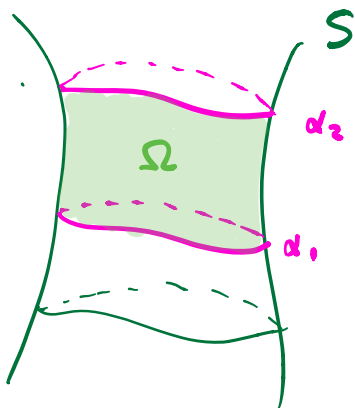
(II) If S is homeomorphic to a cylinder and

$K < 0$ everywhere on S

then $\exists$ at most 1 simple closed geodesic on S

Sketch of Proof: Suppose NOT. We have 2 such geodesics

α_1, α_2 which together must bound a cylinder in S



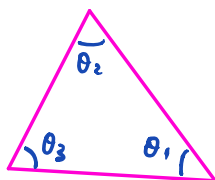
Apply Gauss-Bonnet II to Ω

$$\underbrace{\int_{\Omega} K da}_{< 0} + \int_{\partial\Omega} \cancel{k_g} ds = 2\pi \cancel{\chi(\Omega)}$$

Contradiction!

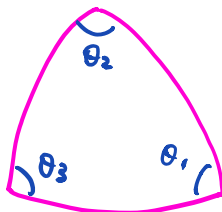
(IV) Interior angle sum of "geodesic triangles"

$K \equiv 0$



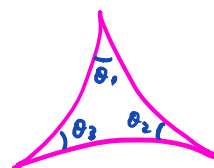
$$\sum \theta_i = \pi$$

$K > 0$



$$\sum \theta_i > \pi$$

$K < 0$



$$\sum \theta_i < \pi$$

§ Proof of Gauss-Bonnet Theorems

We will prove Gauss-Bonnet Theorem I, i.e.

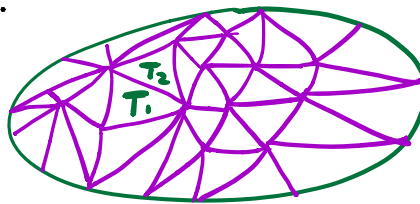
$$\int_S K \, da = 2\pi \chi(S)$$

for any compact
orientable surface S
without boundary

First, we reduce the problem to the "local" case by

FACT: Any orientable surface can be "triangulated".

eg.



$$S = \bigcup_i T_i$$

By subdivision we can assume that each T_i is small enough that it is contained in a single coordinate neighborhood.

FACT: Any compact orientable surface can be covered by "conformal coordinate systems":

i.e. $X: \mathcal{U} \subset \mathbb{R}^2 \rightarrow S$

s.t. $(g_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$
1st f.f.

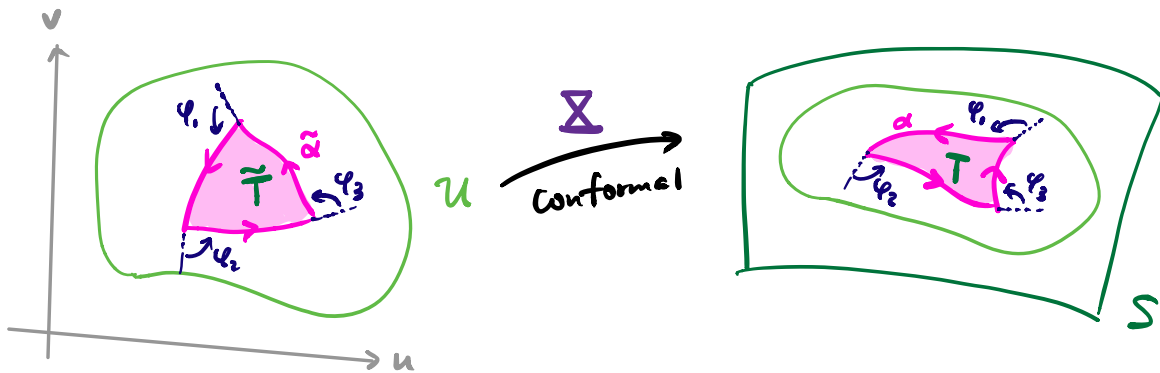
for some smooth
positive function
 $\lambda: \mathcal{U} \rightarrow \mathbb{R}$.

Note: Conformal coordinates preserve infinitesimal angles!

We will prove the following local Gauss-Bonnet Theorem:

Local Gauss-Bonnet:

Let $\Sigma : U \subseteq \mathbb{R}^2 \rightarrow S$ be a conformal parametrization.



$$\int_T K \, da + \int_{\partial T} k_g \, ds + \sum_i \varphi_i = 2\pi$$

Proof: In conformal coordinates,

$$k_g = \frac{1}{2\lambda} \left(\frac{\partial \lambda}{\partial u} v' - \frac{\partial \lambda}{\partial v} u' \right) + k$$

where k = curvature of $\tilde{\alpha}$ (as a plane curve in $\mathbb{R}^2$)

$\tilde{\alpha}(s) = (u(s), v(s))$ p.b.a.l. as a plane curve

$$\begin{aligned}
 \int_{\partial T} k_g ds + \sum_i \varphi_i &= \int_{\tilde{\alpha}} \frac{1}{2\lambda} \left(\frac{\partial \lambda}{\partial u} v' - \frac{\partial \lambda}{\partial v} u' \right) \\
 &\quad + \underbrace{\int_{\tilde{\alpha}} k + \sum_i \varphi_i}_{2\pi \text{ (Thm. of Turning tangent)}}
 \end{aligned}$$

By Green's Theorem,

$$\begin{aligned}
 \int_{\tilde{\alpha}} \frac{1}{2\lambda^2} \left(\frac{\partial \lambda^2}{\partial u} v' - \frac{\partial \lambda^2}{\partial v} u' \right) &= \iint_{\tilde{T}} \frac{\partial}{\partial u} \left(\frac{1}{2\lambda} \frac{\partial \lambda}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{1}{2\lambda} \frac{\partial \lambda}{\partial v} \right) \\
 &= \iint_{\tilde{T}} \frac{1}{2} \Delta (\log \lambda) du dv \\
 &= \iint_{\tilde{T}} \underbrace{\frac{1}{2\lambda} \Delta (\log \lambda)}_{-K} \underbrace{\lambda du dv}_{da} \\
 &= - \int_{\tilde{T}} K da
 \end{aligned}$$

We now apply local Gauss-Bonnet to each T_i in a "fine" triangulation with

$$\left\{ \begin{array}{l} F = \# \text{ faces} \\ E = \# \text{ edges} \\ V = \# \text{ vertices} \end{array} \right.$$

Recall:

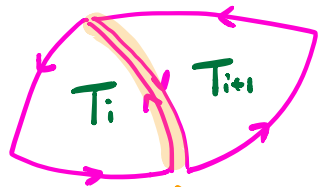
$$\chi(S) = F - E + V$$

$$\int_{T_i} K da + \int_{\partial T_i} k_g ds + \sum_j \varphi_j^i = 2\pi$$

Sum over all the T_i 's, we get

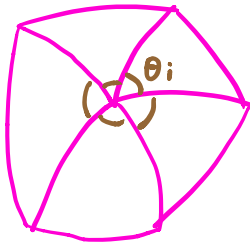
$$\int_S K da + 0 + \frac{3\pi F}{-2\pi V} = 2\pi F$$

Reason: (I)



↑ different orientations $\Rightarrow$ cancels

(II)



$$\text{ext. } \angle \text{ sum} = 3\pi - \text{int. } \angle \text{ sum}$$

$\Downarrow$

$$\sum_{i,j} \varphi_j^i = 3\pi F - 2\pi V$$

$$\text{Hence, } \int_S K da = 2\pi F - \underbrace{3\pi F}_{2\pi E} + 2\pi V$$

$$\parallel$$

$$2\pi E \quad \text{since } 3F = 2E$$

_____ $\circ$